NATIONAL ECONOMICS UNIVERSITY

FACULTY OF MATHEMATICAL ECONOMICS

BACHELOR THESIS

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CASH-FLOW MATCHING:
A Linear Programming Duality Approach

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Cash-flow matching and Bond immunization are two types of dedication strategies, they are amongst the most important and practical methods for managing liquidity risk. The fundamental objective of cash-flow matching is to align cash outflows (liability obligations) with corresponding cash inflows within a specified time frame. The business concern lies not in the value of our bond portfolio, but rather in ensuring that they possess sufficient cash flows to fulfill future liabilities. This paper specifically focuses on a cash-flow matching strategy that utilizes bond principal and coupon payments to achieve the desired outcome. From that, an optimized portfolio will be constructed so that the total cash flows will exactly match the liabilities amount. The duality theory of linear programming will be applied, providing insights for solving and generalizing the cash-flow matching strategy. The process of constructing the least expensive possible portfolio will be discussed further in this paper.

Objectives

Systematize the fundamental theories for further research, including cash-flow matching, liquidity risk, interest rate risk, and linear programming.

Provide more understanding about the dedication strategy: Cash-flow matching using Linear programming duality. In terms of practical applications, the theory of linear programming is one of the most beneficial advances in mathematics. By formulating the dual of a linear problem, the cash-flow matching model can be viewed from a different perspective.

Apply the theoretical knowledge to formulate an actual linear programming cash-flow model for businesses. This paper shows that the action of solely matching the cash inflows with liabilities will give a feasible solution that satisfies real-world scenarios, but the portfolio might be expensive. However, when relaxing more conditions to the cash-flow matching model, the applicability of the model formulation and result will be improved.

Method of Research

The cash-flow matching model obtained from this research will be derived using the theory of Duality in Linear Programming. Approaching the cash flow matching problem through the lens of linear programming, we can utilize the duality theorem to generate alternative formulations of the problem. This enables us to explore various adaptations of the same problem, potentially leading to more applicable solutions.

The feasible solutions in this thesis are calculated with the Solver tool in Microsoft Excel.

Scope of Research

The primary emphasis of this thesis centers on analyzing the cash flow data of MB Bank's liabilities within the timeframe spanning from 2013 to 2022. Additionally, the thesis incorporates data pertaining to Vietnam Treasury Bonds as a supplementary component.

The Structure of the Thesis

The outline of this thesis follows below:

Part 1 will introduce the overall knowledge of liquidity risk, bond immunization, and cash-flow matching that support understanding extended terms in the following parts.

Part 2 will highlight the background of the classical theory of Linear programming Duality as well as the assumptions and ideas to formulate the linear model.

Part 3 will show the formulation of the cash-flow matching model with the term structure of interest rate, carry-forward allowed, and borrowing also allowed.

Part 4 will apply assumptions and formulas from above in the real-world scenario to construct the least expensive portfolio.

LITERATURE REVIEW

I.1 Liquidity risk

I.1.1 Definition of Liquidity risk

Liquidity risk is a concept that plays a crucial role in finance and is of significant concern to individuals, organizations, and financial institutions alike. It refers to the potential for encountering difficulties in meeting short-term financial obligations due to a lack of available cash or easily convertible assets. In essence, liquidity risk arises from the possibility of encountering unforeseen circumstances or events that impede the ability to access funds promptly and at a reasonable cost.

In the dynamic and ever-changing financial landscape, maintaining adequate liquidity is essential for the smooth functioning of economic activities. It enables individuals and organizations to meet their financial commitments, seize investment opportunities, and effectively manage their day-to-day operations. However, liquidity risk poses a constant challenge, as disruptions in the financial markets, economic downturns, or sudden changes in investor sentiment can quickly erode liquidity and create financial vulnerabilities.

One of the primary drivers of liquidity risk is the imbalance between cash inflows and outflows. When an entity's cash outflows exceed its cash inflows, it may face difficulties in fulfilling its financial obligations. For example, a business that relies on consistent cash flow from its operations to cover expenses and debt payments may face liquidity issues if its revenue decreases or its operating costs rise unexpectedly. Similarly, an individual who depends on a regular paycheck to meet daily expenses may experience liquidity constraints if they lose their job or encounter unforeseen expenses.

Another significant factor contributing to liquidity risk is the availability and cost of funding sources. Entities rely on various sources of funding, such as loans, credit lines, and capital markets, to meet their liquidity needs. However, disruptions in the financial markets or a loss of confidence among lenders can significantly limit the availability of funding or increase its cost. This can create challenges for businesses and individuals seeking to access additional liquidity during times of financial stress. Furthermore, liquidity risk can be exacerbated by market conditions and the specific characteristics of financial instruments. In times of market volatility or financial crises, asset prices can decline rapidly, making it challenging to sell assets quickly without incurring significant losses. Illiquid or hard-to-value assets, such as certain types of securities or complex derivatives, can further amplify liquidity risk as they may be challenging to trade or value accurately in times of stress.

Managing liquidity risk requires a comprehensive approach that involves understanding and monitoring cash flow patterns, maintaining adequate reserves of liquid assets, and establishing contingency plans to address potential liquidity shortfalls. Financial institutions employ various liquidity risk management tools and strategies to ensure their ongoing solvency and ability to meet their obligations. These include stress testing, scenario analysis, liquidity risk limits, and diversification of funding sources.

In conclusion, liquidity risk is an essential aspect of financial management that encompasses the potential for encountering difficulties in meeting short-term financial obligations. It arises from imbalances between cash inflows and outflows, disruptions in funding sources, and market conditions. Managing liquidity risk is crucial for maintaining financial stability, ensuring the ability to meet obligations, and navigating through turbulent economic environments. By understanding and effectively managing liquidity risk, individuals, organizations, and financial institutions can enhance their resilience and safeguard their financial well-being.

I.1.2 Impact of Liquidity risk on Bond prices

Liquidity risk can have a significant impact on bond prices, influencing their valuation and attractiveness in the market. When liquidity conditions deteriorate, it becomes more challenging to buy or sell bonds at desired prices, leading to changes in supply and demand dynamics. Here are some key impacts of liquidity risk on bond prices:

- Price Discounts: During periods of heightened liquidity risk, investors may become more cautious and demand higher compensation for taking on the risk of holding less liquid bonds. As a result, bond prices tend to decline, leading to price discounts. Investors may require a lower purchase price to compensate for the uncertainty and potential difficulty in selling the bonds in the future.

- Increased Yield Spreads: Liquidity risk can also widen yield spreads, which represent the additional yield investors demand for holding a particular bond compared to a benchmark, such as government bonds. Higher liquidity risk reduces market depth and increases transaction costs, leading to wider bid-ask spreads and, consequently, higher yields for less liquid bonds.
- Reduced Market Depth: Liquidity risk can negatively impact market depth, which refers to the ability to buy or sell bonds in significant quantities without causing substantial price movements. During periods of liquidity stress, market participants may become reluctant to take on large positions, leading to reduced liquidity and thinner trading volumes. This can result in larger price swings and increased volatility.
- Flight to Quality: When liquidity conditions deteriorate, investors tend to seek safer and more liquid assets. This flight to quality can lead to a decline in demand for bonds with higher liquidity risk, causing their prices to decrease. Investors may prioritize bonds issued by stable governments or highly rated corporations, as these are perceived as more liquid and less likely to be affected by liquidity disruptions.
- Liquidity Premium: Bonds with higher liquidity risk may have a liquidity premium embedded in their yields. Investors require compensation for the potential challenges in buying or selling these bonds, resulting in higher yields compared to similar bonds with lower liquidity risk. The liquidity premium reflects the additional return investors demand for holding a less liquid asset.
- Market Sentiment Effects: Liquidity risk can also influence investor sentiment and market psychology. If concerns about liquidity conditions prevail in the market, it can lead to a general decrease in confidence and a shift towards more risk-averse behavior. This can further impact bond prices as investors become more cautious and demand higher compensation for holding bonds with perceived liquidity risk.

It is important to note that the impact of liquidity risk on bond prices can vary depending on the specific characteristics of the bond, market conditions, and the severity of the liquidity stress. Different types of bonds may exhibit varying degrees of liquidity risk, with less liquid bonds typically experiencing more pronounced price impacts during periods of liquidity stress.

Overall, liquidity risk has the potential to significantly influence bond prices, leading to price discounts, wider yield spreads, reduced market depth, and the emergence of a liquidity premium. Investors and market participants closely monitor liquidity conditions and incorporate liquidity risk considerations into their investment decisions, particularly when assessing the value and risk-reward profile of bonds in the market.

I.2 The term structure of Interest rate

The term structure of interest rates refers to the relationship between the interest rates and the time to maturity of debt instruments, such as bonds or loans. It depicts the prevailing interest rates for different maturities, ranging from short-term to long-term. The term structure is typically represented graphically using a yield curve. A yield curve is a line that plots the interest rates (yields) on the vertical axis against the corresponding maturities on the horizontal axis. The shape of the yield curve provides insights into market expectations, investor sentiment, and economic conditions.

There are several common shapes of yield curves:

- Normal Yield Curve: In a normal yield curve, longer-term interest rates are higher than short-term rates. This shape reflects the expectation that the economy will grow over time, leading to higher inflation and higher interest rates in the future.
- Inverted Yield Curve: An inverted yield curve occurs when short-term interest rates are higher than long-term rates. This shape is often seen as a predictor of an economic downturn or recession. Investors may expect central banks to lower interest rates in the future due to economic weakness.
- Flat Yield Curve: A flat yield curve indicates that short-term and long-term interest rates are relatively similar. This shape suggests uncertainty about future economic conditions or interest rate movements.
- Steep Yield Curve: A steep yield curve occurs when there is a significant difference between short-term and long-term interest rates. This shape may indicate expectations of economic growth and potential inflationary pressures in the future.

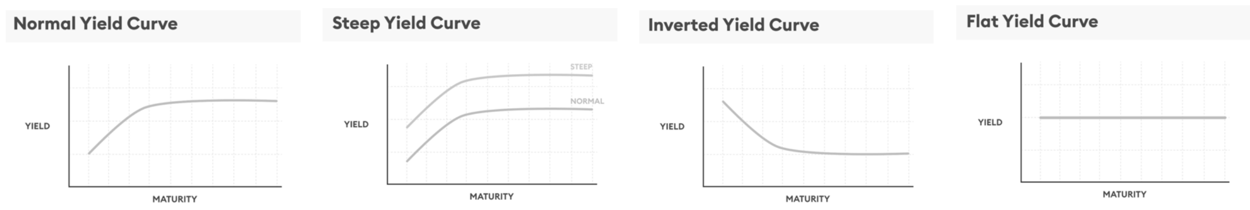


Figure I.2.1 : Four types of Yield Curve

It's important to note that the shape of the yield curve can change over time based on various factors such as central bank policies, market conditions, inflation expectations, and investor demand for different maturities.

I.3 Cash-flow matching

I.3.1 Definition of Cash-flow matching

Cash flow matching is a financial strategy that involves aligning the cash inflows and outflows of an entity or investment portfolio to ensure that the timing and amounts of cash flows match or closely correspond to each other. The goal of cash flow matching is to minimize the risk of cash flow mismatches and ensure that there are sufficient funds available to meet financial obligations when they arise.

In cash flow matching, the cash flows are typically matched to specific liabilities or payment obligations. This is commonly done by investing in assets or securities that generate cash flows at specific times, such as bonds or fixed-income instruments with similar maturity dates to the liabilities. By matching the cash flows in this way, the entity or portfolio aims to meet its obligations without relying on unpredictable market conditions or having to sell assets at unfavorable prices.

Cash flow matching is commonly used in various financial contexts, including pension funds, insurance companies, and project financing. It helps these entities manage their cash flow obligations, mitigate risk, and ensure a stable and predictable stream of funds to fulfill their commitments.

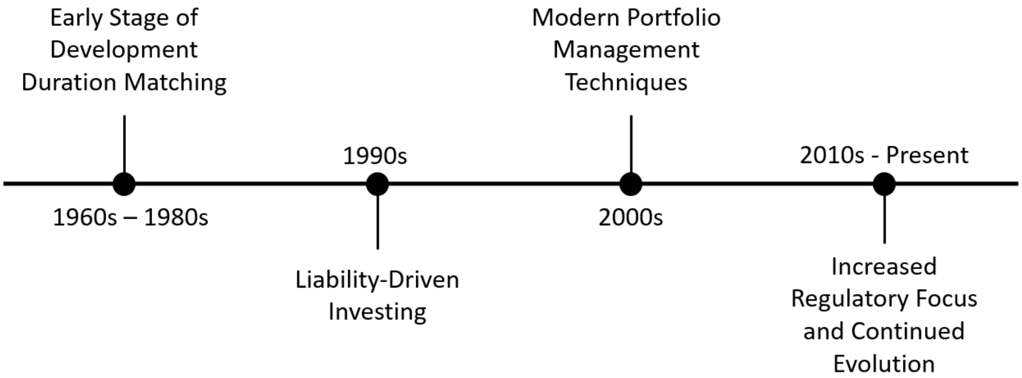


Figure I.3.1 : The Development history of Cash-flow matching techniques

I.3.2 The development process of Cash-flow matching

The development of the cash-flow matching technique has evolved over time, driven by advancements in financial theory and the need for more sophisticated liability management strategies. Here is a brief overview of the historical development of cash-flow matching:

1. Early Stages (1960s - 1980s): The origins of liability-driven investing can be traced back to the 1960s and 1970s when pension funds and insurance companies began to recognize the importance of matching their long-term payment obligations with appropriate assets. Traditional asset allocation approaches focused on maximizing returns rather than considering the timing and nature of liabilities. The concept of duration, a measure of the sensitivity of bond prices to changes in interest rates, gained prominence in the 1980s. Duration matching became a key component of cash-flow matching, as investors realized the importance of aligning the duration of assets with that of liabilities to minimize interest rate risk.

2. Liability-Driven Investing (1990s): In the 1990s, liability-driven investing (LDI) emerged as a comprehensive approach to managing the assets and liabilities of institutional investors. LDI incorporated the principles of cash-flow matching and duration matching, but with a more holistic focus on liability management and risk mitigation.

3. Modern Portfolio Management Techniques (2000s): With advancements in financial modeling and computational capabilities, the 2000s saw the integration of modern portfolio management techniques into cash-flow matching strategies. These techniques in-

cluded optimization algorithms, risk factor analysis, and scenario testing to enhance the precision and efficiency of portfolio construction and risk management.

4. Increased Regulatory Focus and Continued Evolution (2010s - Present): The 2010s brought increased regulatory scrutiny and risk management requirements for institutional investors. Regulatory bodies, such as the Pension Protection Act in the United States and Solvency II in Europe, emphasized the need for prudent liability management practices. This further fueled the adoption of cash-flow matching and LDI strategies. Cash-flow matching techniques continue to evolve in response to changing market conditions, investor demands, and regulatory landscapes. The integration of alternative investments, dynamic asset allocation strategies, and advancements in liability modeling are areas of ongoing development in the field of liability-driven investing.

The primary focus of this paper will be the examination of the initial phases of Cash-flow matching, as they serve as the foundation of contemporary cash-flow models. The underlying rationale and logic behind these cash-flow matching techniques will be extensively explored and discussed within the confines of this thesis.

I.3.3 The importance of Cash-flow matching in Businesses

Cash flow matching enables businesses to effectively manage their liabilities by aligning cash inflows with cash outflows. By matching the timing and amount of cash flows, businesses can ensure they have the necessary funds available to meet their financial obligations as they arise. This helps prevent cash flow mismatches and reduces the risk of financial distress or default. Cash flow matching helps mitigate risks associated with uncertain cash flows. By matching cash inflows with specific liabilities or payment obligations, businesses reduce their reliance on unpredictable market conditions or the need to sell assets at unfavorable prices to generate funds. This provides stability and predictability in meeting financial obligations, reducing the impact of market volatility or unexpected events.

Cash flow matching is important for effective funding and capital management. By aligning cash inflows and outflows, businesses can optimize their cash management strategies, minimize the need for external financing or borrowing, and ensure efficient allocation of capital. This improves liquidity and reduces reliance on costly financing options, leading to better financial performance. Cash flow matching is essential for investment planning and decision-making. By considering the timing and amount of expected cash inflows and outflows, businesses can make informed investment choices that align with their cash flow requirements. This helps optimize the allocation of resources, maximize returns, and minimize the risk of illiquid investments.

Cash flow matching contributes to the long-term sustainability of businesses. By ensuring a stable and predictable cash flow stream, businesses can maintain financial stability, meet their obligations, and pursue growth opportunities. It provides a solid foundation for business planning, risk management, and maintaining healthy financial positions.

Overall, cash flow matching enhances financial discipline, reduces risk, and improves the financial health and stability of businesses. It enables effective liability management, risk mitigation, funding optimization, investment planning, and long-term sustainability, making it a critical component of successful business operations.

LINEAR PROGRAMMING DUALITY

II.1 Introduction to the Dual of Linear Programming

The Duality theory of linear programming serves as the primary computation method in this paper. Linear programming problems are optimization problems in which the objective function and the constraints are all linear. The objective function in the primal problem is a linear combination of n variables. A linear combination of the n variables has an upper bound set by each of the m conditions, which means there are m constraints. The purpose is to maximize (or minimize) the value of the objective function under the constraints. A vector of n values that achieves the objective function's maximum (or minimum) value is referred to as a feasible or optimal solution. We now going to briefly highlight the duality theory of linear programming.

Linear programming duality is an important concept in optimization theory. It establishes a fundamental relationship between two forms of linear programming problems known as the primal problem and the dual problem.

In linear programming, the primal problem involves maximizing or minimizing a linear objective function while satisfying a set of linear constraints. The variables in the primal problem represent the decision variables that need to be determined.

The dual problem, on the other hand, is derived from the primal problem and provides an alternative perspective on the same optimization task. It involves minimizing or maximizing a different linear objective function, subject to a set of constraints. The variables in the dual problem represent the dual variables, also known as shadow prices or dual prices, which have interpretations related to the constraints of the primal problem. The duality theory in linear programming establishes that under certain conditions, the optimal solution to the primal problem is linked to the optimal solution of the dual problem. This relationship is captured by the duality theorem, which states that the optimal objective values of the primal and dual problems are equal.

Additionally, the duality theorem implies that the dual problem can provide valuable insights and information about the primal problem. The dual variables or shadow prices associated with the constraints in the dual problem can provide economic interpretations, such as representing the marginal value or cost of resources in the primal problem.

The duality concept extends to several properties and relationships, such as complementary slackness, which states that if a constraint in the primal problem is binding (holds with equality), then the corresponding dual variable is positive, and vice versa. The duality concept in linear programming is extensively used in various applications, including resource allocation, production planning, transportation, and scheduling problems. It enables the understanding of both the primal and dual perspectives of an optimization problem, offering valuable insights and solution approaches.

II.1.1 Definition of Linear programming

This section aims to provide an introduction to the concept of Linear Programming Duality, drawing upon the knowledge acquired from the subject "Optimization" taught by Mrs. Nguyen Thi Thao. The definition and fundamental principles of Linear Programming Duality will be discussed. Imagine that we have to solve a linear program and an LP solver has found us the solution. However, we will not know if the solution is indeed optimal for the linear problem without having to trace back steps of the algorithm's computation.

Therefore, the dual of the primal problem is created by scaling and setting bounds to prove the optimum, and that the feasible solutions of both primal and dual problem are the same.

Suppose we have a primal maximization linear program

$$
\begin{aligned}
&\text{Maximize } c_1x_1 + c_2x_2 + ... + c_nx_n \\
&\text{subject to} \\
&a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,n}x_n \leq b_1 \\
&a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,n}x_n \leq b_2 \\
&\vdots \\
&a_{m,1}x_1 + a_{m,2}x_2 + ... + a_{m,n}x_n \leq b_m \\
&x_1 \geq 0, ..., x_n \geq 0
\end{aligned}
$$

we can derive the inequality with every choice of non-negative scaling factors $y_1, y_2, ..., y_m$

$$
\begin{aligned}
&y_1(a_{1,1}x_1 + a_{1,2}x_2 + ... + a_{1,n}x_n)+\\
&y_2(a_{2,1}x_1 + a_{2,2}x_2 + ... + a_{2,n}x_n)+\\
&... + ...\\
&y_m(a_{m,1}x_1 + a_{m,2}x_2 + ... + a_{m,n}x_n)\\
&\leq y_1b_1 + y_2b_2 + ... + y_mb_m
\end{aligned}
$$

we will rewrite the inequality as

$$
\begin{aligned}
&(a_{1,1}y_1 + a_{2,1}y_2 + ... + a_{m,1}y_m)x_1+\\
&(a_{1,2}y_1 + a_{2,2}y_2 + ... + a_{m,2}y_m)x_2+\\
&... + ...\\
&(a_{1,n}y_1 + a_{2,n}y_2 + ... + a_{m,n}y_m)x_n\\
&\leq y_1b_1 + y_2b_2 + ... + y_mb_m
\end{aligned}
$$

we now choose y_i so that the linear function of x_i for which we get an upper bound is, in turn, an upper bound to the cost function of $(x_1, x_2, ..., x_n)$

$$
\begin{aligned}
&c_1 \leq a_{1,1}y_1 + a_{2,1}y_2 + ... + a_{m,1}y_m\\
&c_1 \leq a_{1,2}y_1 + a_{2,2}y_2 + ... + a_{m,1}y_m\\
&\vdots\\
&c_1 \leq a_{1,n}y_1 + a_{2,n}y_2 + ... + a_{m,n}y_m
\end{aligned}
$$

we see that

$$
\begin{aligned}
&c_1x_1 + c_2x_2 + ... + c_nx_n\\
&\leq (a_{1,1}y_1 + a_{2,1}y_2 + ... + a_{m,1}y_m)x_1+\\
&(a_{1,2}y_1 + a_{2,2}y_2 + ... + a_{m,2}y_m)x_2+\\
&... + ...\\
&(a_{1,n}y_1 + a_{2,n}y_2 + ... + a_{m,n}y_m)x_n\\
&\leq y_1b_1 + y_2b_2 + ... + y_mb_m
\end{aligned}
$$

we want to find the non-negative values $y_1, y_2, ..., y_m$ such that the above upper bound is as strong as possible, therefore the objective function is

$$\begin{aligned}
&\text{Minimize } b_1y_1 + b_2y_2 + ... + b_my_m \\
&\text{subject to} \\
&a_{1,1}y_1 + a_{2,1}y_2 + ... + a_{m,1}y_m \geq c_1 \\
&a_{1,2}y_1 + a_{2,2}y_2 + ... + a_{m.1}y_m \geq c_2 \\
&\vdots \\
&a_{1,n}y_1 + a_{2,n}y_2 + ... + a_{m,n}y_m \geq c_n \\
&y_1 \geq 0, ..., y_m \geq 0
\end{aligned}$$

we discover that if we want to find the scaling factors that give us the best possible upper bound for a primal maximization linear problem, we end up with the dual minimization linear problem.

Definition 1 Let m and n be positive integers, and let $\boldsymbol{A} = (\boldsymbol{a_{i,j}})$ be a real m by n matrix. Let $\boldsymbol{b}$ and $\boldsymbol{c}$ be (column) vectors in $\boldsymbol{R^m}$ and $\boldsymbol{R^n}$ respectively. Let $\mathbf{0}$ denotes the zero vectors, the dimension of the zero vectors will be determined by the context.
If

$$\begin{aligned}
\text{Maximize } \boldsymbol{c^T x} \\
\text{subject to } \boldsymbol{Ax} \leq \boldsymbol{b} \\
\boldsymbol{x} \geq \mathbf{0}
\end{aligned}$$

is the primal linear problem in standard form, then its dual problem will be

$$\begin{aligned}
\text{Minimize } \boldsymbol{b^T y} \\
\text{subject to } \boldsymbol{A^T y} \geq \boldsymbol{c} \\
\boldsymbol{y} \geq \mathbf{0}
\end{aligned}$$

The primal linear programming problem is to find the feasible vector $\boldsymbol{x} \geq \mathbf{0}$ in $\boldsymbol{R^n}$ which satisfies the system of m linear inequalities $\boldsymbol{Ax} \leq \boldsymbol{b}$ and also maximizes the objective function

$$\boldsymbol{c^T x} = c_1x_1 + c_2x_2 + ... + c_nx_n$$

The dual problem seeks to determine the dual feasible vector $\boldsymbol{y} \geq \boldsymbol{0}$ in $\boldsymbol{R^m}$ that satisfies the system of n linear inequalities $\boldsymbol{A^T y} \geq \boldsymbol{c}$ and minimizes the objective function

$$\boldsymbol{b^T y} = b_1 y_1 + b_2 y_2 + ... + b_m y_m$$

Definition 2 The previous definition starts with the primal problem in the maximization form, we now do the same thing but beginning with a minimization linear program.
We have the primal problem

$$\begin{aligned} \text{Minimize } & \boldsymbol{b^T y} \\ \text{subject to } & \boldsymbol{A^T y} \geq \boldsymbol{c} \\ & \boldsymbol{y} \geq \boldsymbol{0} \end{aligned}$$

which is equivalent to

$$\begin{aligned} \text{Maximize } & -\boldsymbol{b^T y} \\ \text{subject to } & -\boldsymbol{A^T y} \leq -\boldsymbol{c} \\ & \boldsymbol{y} \geq \boldsymbol{0} \end{aligned}$$

if we take the dual problem of the above problem we get

$$\begin{aligned} \text{Minimize } & -\boldsymbol{c^T x} \\ \text{subject to } & -\boldsymbol{Ax} \geq -\boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

which is *the dual of the primal minimization program*

$$\begin{aligned} \text{Maximize } & \boldsymbol{c^T x} \\ \text{subject to } & \boldsymbol{Ax} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

In general, we can form the dual of a minimization linear program in the same way of forming a maximization linear program's dual problem.

1. First step is to switch the type of optimization, from minimizing to maximizing and vice versa.

2. Second step is to define the dual variables and constraints: the number of dual problem's variables must match the number of primal problem's constraints, and the number of dual problem's constraints must match the number of primal problem's variables (non-negativity constraints are not included). The second step can also be understood as the dual is formed by having one variable for each constraint and one constraint for each variable.
3. Next step, we need to take the transpose of the coefficient matrix on the left-hand side of the inequality.
4. Finally, switch the roles of the coefficient vector in the objective function and the vector of right-hand sides in the inequalities.

II.1.2 Theorem of Linear programming Duality

Theorem 1 The dual of the dual of a linear program is the linear program itself.

Theorem 2 If the primal (in maximization standard form) and the dual (in minimization standard form) are both feasible, then

$$optimal(primal) \leq optimal(dual)$$

Proof: *Whenever* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are feasible,*

$$\boldsymbol{c^T x} \leq \boldsymbol{(A^T y)^T x} \implies \boldsymbol{c^T x} \leq \boldsymbol{(y^T A) x} \implies \boldsymbol{c^T x} \leq \boldsymbol{y^T (Ax)}$$

$$\text{we have } \boldsymbol{y^T (Ax)} \leq \boldsymbol{y^T b}$$

$$\implies \boldsymbol{c^T x} \leq \boldsymbol{y^T b}$$

consequently, we can write ***Theorem 2*** as

$$\sup \{\boldsymbol{c^T x} | x \geq 0 \text{ in } R^n \text{ and } \boldsymbol{Ax \leq b}\} \leq \inf \{\boldsymbol{b^T y} | y \geq 0 \text{ in } R^m \text{ and } \boldsymbol{A^T y \geq c}\}$$

where an empty supremum equals $-\infty$ and an empty infimum equals $+\infty$

Theorem 3 If the primal (in minimization standard form) and the dual (in maximization standard form) are both feasible, then

$$opt(primal) \geq opt(dual)$$

Proof: *Whenever* $\boldsymbol{x}$ *and* $\boldsymbol{y}$ *are feasible,*

$$\boldsymbol{b^T y} \geq (\boldsymbol{Ax})^{\boldsymbol{T}}\boldsymbol{y} \implies \boldsymbol{b^T y} \geq (\boldsymbol{x^T A^T})\boldsymbol{y} \implies \boldsymbol{b^T y} \geq \boldsymbol{x^T}(\boldsymbol{A^T y})$$

$$\text{we have } \boldsymbol{x^T}(\boldsymbol{A^T y}) \leq \boldsymbol{x^T c}$$

$$\implies \boldsymbol{b^T y} \geq \boldsymbol{x^T c}$$

consequently, we can write ***Theorem 3*** as

$$\inf \{\boldsymbol{b^T y} | y \geq 0 \text{ in } R^n \text{ and } \boldsymbol{A^T y} \geq \boldsymbol{c}\} \geq \sup \{\boldsymbol{c^T x} | x \geq 0 \text{ in } R^m \text{ and } \boldsymbol{Ax} \leq \boldsymbol{b}\}$$

where an empty supremum equals $-\infty$ and an empty infimum equals $+\infty$

Theorem 4 The celebrated fundamental theorem of linear programming states that the inequalities in ***Theorem 2*** and ***Theorem 3*** are in fact an equality:

$$\sup \{\boldsymbol{c^T x} | x \geq 0 \text{ in } R^n \text{ and } \boldsymbol{Ax} \leq \boldsymbol{b}\} = \inf \{\boldsymbol{b^T y} | y \geq 0 \text{ in } R^m \text{ and } \boldsymbol{A^T y} \geq \boldsymbol{c}\}$$

and

$$\inf \{\boldsymbol{b^T y} | y \geq 0 \text{ in } R^n \text{ and } \boldsymbol{A^T y} \geq \boldsymbol{c}\} = \sup \{\boldsymbol{c^T x} | x \geq 0 \text{ in } R^m \text{ and } \boldsymbol{Ax} \leq \boldsymbol{b}\}$$

unless both primal and dual problems are infeasible.

Theorem 5 An equality $r = s$ is equivalent to the pair of simultaneous inequalities $r \leq s$ and $-r \leq -s$. Moreover, each real number r can be written as the difference of two non-negative numbers, $r = r^+ - r^-$ with $r^+ \geq 0$, $r^- \geq 0$. Hence the fundamental theorem of linear programming can be modified as:

$$\begin{aligned} &\sup \{\boldsymbol{c^T x} | x \text{ unconstrained in sign in } R^n \text{ and } \boldsymbol{Ax} \leq \boldsymbol{b}\} \\ &\qquad = \inf \{\boldsymbol{b^T y} | y \geq 0 \text{ in } R^m \text{ and } \boldsymbol{A^T y} \geq \boldsymbol{c}\} \end{aligned}$$

unless both primal and dual problems are infeasible.

II.2 Cash-flow matching using Linear Programming Duality

The aim of creating this model is quite straight-forward, it is to ensure that the business will always be able to pay its liability obligations and avoid being default. Suppose that a business has a stream of liability obligations - denoted as l_t at time t. The liability l_t is to be paid at time t, $t = 1, 2, 3, ...$ The liability cash flows - denoted as $\{l_t\}$ are assumed to be fixed and certain. And for simplicity, we will also assume that all cash flows occur at the end of time periods.

Assume that the bonds are non-callable and default-free fixed-income securities. The business must construct an investment portfolio that will meet the future liability obligations, therefore, the initial investment portfolio needs to have a cash flow that meets the projected liability payment for each and every time period in the planning horizon. Furthermore, the cost of that portfolio must be as low as possible.

We use the following notation:
p_k - the current price of one unit of the k-th security
n_k - the number of units of the k-th security to be purchased
$c_{k,t}$ - the cash flow of security k at time t, t = 1, 2, 3, ...
l_t - the liability to be paid at time t, t = 1, 2, 3, ...

The tables below will demonstrate how the portfolio will look like:

Security	**Bond price**	**CF at time 1**	**CF at time 2**		**CF at time t**
Bond 1	p_1	$c_{1,1}$	$c_{1,2}$	...	$c_{1,t}$
Bond 2	p_2	$c_{2,1}$	$c_{2,2}$	...	$c_{2,t}$
...	...	...	...	...	...
Bond k	p_k	$c_{k,1}$	$c_{k,2}$	...	$c_{k,t}$

Table II.2.1 : Cash flows from buying securities

Security	**Units**	**Cost**	**CF at time 1**	**CF at time 2**		**CF at time t**
Bond 1	n_1	n_1p_1	$n_1c_{1,1}$	$n_1c_{1,2}$	...	$n_1c_{1,t}$
Bond 2	n_2	n_2p_2	$n_2c_{2,1}$	$n_2c_{2,2}$	...	$n_2c_{2,t}$
...	...	...	...	...	...	...
Bond k	n_k	n_kp_k	$n_kc_{k,1}$	$n_kc_{k,2}$	...	$n_kc_{k,t}$

Table II.2.2 : Bond Portfolio construction

From the information above, we can see that the business wants to find the optimal investment portfolio $\{n_k\}$ by minimizing the total cost, keep in mind that n is non-negative

$$\sum_{n=1}^{k} n_k p_k$$

under the constraints

$$\sum_{n=1}^{k} n_k c_{k,t} \geq l_t, \text{ for all } t$$

and

$$n_k \geq 0, \text{ for all } t$$

Overall ideas of finding the best investment portfolio is relatively simple, however, the strict condition of matching cash inflows and cash outflows perfectly can lead to expensive portfolios. The linear programming formulation for the optimal portfolio of this problem will be discussed further in the paper.

Based on the problem above, in order to implement the cash-flow matching technique, the business only needs to know the information about their future cash flows and the prices of the fixed-income bonds available in the market. One of the primary benefits of using this cash-flow matching technique is its simplicity, as the company need not worry about the length, convexity, or term structure of the interest rate. However, the cash-flow matching technique actually relies on the interest rate's term structure. The concept of interest rate's term structure arises naturally as we take the cash-flow linear program's dual problem. This paper will show a useful extension of the classic linear program formulation through the use of the duality theorem, from that, emphasizing the intrinsic role of the term structure of interest rate.

DEVELOPMENT OF THE CASH-FLOW MATCHING MODEL

III.1 The term structure of Interest rate

Previous section has shown the basic knowledge of linear programming duality as well as the formulation of the cash-flow matching problem. To start off, the first model will be uncomplicated as it is based solely on the business' intention of keeping the cash inflows greater or equal than cash outflows. To apply the duality theorem, the linear program **LP1** can be written as

$$\text{Minimize } \boldsymbol{p^T n} \tag{1}$$

subject to

$$\boldsymbol{C^T n \geq l} \tag{2}$$

where

$$p = (p_1, p_2, ..., p_k)^T$$
$$n = (n_1, n_2, ..., n_k)^T$$
$$l = (l_1, l_2, ..., l_t)^T$$
$$C = (c_{i,j})$$

The dual problem of **LP1** - denoted as **LP*1** is

$$\text{Maximize } \boldsymbol{l^T v}$$
$$\boldsymbol{v \geq 0} \tag{3}$$

subject to

$$\boldsymbol{Cv \leq p} \tag{4}$$

The dual problem **LP*1** introduces us to a new variable $\boldsymbol{v}$. The interpretation of $\boldsymbol{v}$ is as follow: Let i_t be the t-period spot rate, for $t = 1, 2, 3, ...$, we have the present value at time 0 for \$1 to be paid at time t is $(1 + t_t)^{-t}$.

The term structure of interest rates is the shape of the graph of i_t and t $(t > 0)$. Assume the business is in a perfect capital market, in which there are no arbitrage opportunities, no taxes, no transaction costs,etc., each non-callable and default-free fixed-income security is priced by the spot rates $\{i_t\}$; that is for each k

$$p_k = \frac{c_{k,1}}{(1+i_1)^1} + \frac{c_{k,2}}{(1+i_2)^2} + ... + \frac{c_{k,t}}{(1+i_t)^t}$$

$$p_k = \sum_{t=1}^{k} \frac{c_{k,t}}{(1+i_t)^t} \tag{5}$$

The equation above is actually the mathematics formula of bond price p_k by cash flows C and spot rate i. In order to satisfy condition (4) of the linear program **LP*1**, we call vector $\boldsymbol{u} = [(1+i_1)^{-1}, (1+i_2)^{-2}, ..., (1+i_t)^{-t}]^T$, which will satisfy the equation $\boldsymbol{Cu} = \boldsymbol{p}$. Because of that, vector $\boldsymbol{u}$ will be the feasible vector of **LP*1**.

By the inequality mentioned in ***Theorem 3*** Section II.1, we have

$$\inf \{\boldsymbol{p^T n} | n \geq 0 \text{ in } R^m \text{ and } \boldsymbol{C^T n \geq l}\} \geq \sup \{\boldsymbol{l^T u} | u \geq 0 \text{ in } R^n \text{ and } \boldsymbol{Cu \leq p}\}$$

Therefore, the objective function of **LP*1** can be written as

$$\boldsymbol{l^T u} = \sum_{t=1}^{k} \frac{l_t}{(1+i_t)^t} \tag{6}$$

and $\boldsymbol{l^T u}$ is the lower bound for the minimum cost (1) of **LP1**. This is an interesting fact since the sum $\boldsymbol{l^T u}$ is exactly the present value of the liability cash flows. The meaning of this can be explained as the minimum cost of the portfolio is always equal to or greater than the *'present value'* of the liability cash flows.

From the interpretation above, it seems that in an economic perspective, vector $\boldsymbol{v}$ which is the feasible solution of **LP*1** is a vector of *discount factors* and the objective function $\boldsymbol{l^T v}$ is the *'present value'* of the liability cash flows.

Let's denote vector $\boldsymbol{v}$ as

$$\boldsymbol{v} = (v_1, v_2, ..., v_t, ...)^T$$

if $\{v_t\}$ are discount factors, we should have the monotonicity condition

$$v_1 \geq v_2 \geq ... \geq v_t ... \geq 0 \tag{7}$$

However, the condition above cannot be satisfied by vector $\boldsymbol{v}$, as we take the example below: Let $l = (7, -3, 6, 8, -4)^T$, $p = (1, 1, 1, 1, 1)^T$ and

$$C^T = \begin{pmatrix} 1.03 & 0.035 & 0.04 & 0.0425 & 0.045 \\ 0 & 1.035 & 0.04 & 0.0425 & 0.045 \\ 0 & 0 & 1.04 & 0.0425 & 0.045 \\ 0 & 0 & 0 & 1.0425 & 0.045 \\ 0 & 0 & 0 & 0 & 1.045 \end{pmatrix}$$

We have the present value of the liability cash flows under the terms structure of interest rates it

$$\boldsymbol{l^T C^{-1} p} = 12.888$$

Note that in the model formulation above, there is no restriction that $\boldsymbol{l} \geq \boldsymbol{0}$, which means the liability cash flows $\{l_t\}$ can be negative. Cash-flow matching model can be used to re-balance an existing portfolio, therefore, negative liability cash flows can be because of currently owned assets that cannot be or are not to be traded. Solving the example, we have the optimal value equals **19.3971** and the optimal solution of **LP*1** is $\boldsymbol{v} = (0.9709, 0, 0.9242, 0.8820, 0)$, though this ***does not satisfy condition (7)***. For more details on the calculation process, please refer to Appendix A.

III.2 Cash-flow matching with Carry-forward allowed

Problems arise as we take the dual linear program of **LP1**, the initial cash-flow matching model in the previous section has shown us its weaknesses. One of them was the absence of the monotonicity condition (7). From an economic point of view, it seems that the constraint $\boldsymbol{C^T n} \geq \boldsymbol{l}$ is unnecessarily restrictive, the model can be extended and allow for the *carry-forward of positive cash balances at zero or low-interest rate.* Which is, by allowing positive cash balances to be reinvested at a conservative reinvestment rate r_t. Surprisingly, if we include the carry-forward feature in the model, condition (7) will automatically be satisfied. It then will follow the ***Theorem 5*** of the fundamental theorem of linear programming that the minimum cost of the portfolio is the same as the maximum 'present value' of the liability cash flows.

This section will describe the formulation of the new cash-flow matching model that allows carry-forward of positive cash balances at zero or low-interest rates. We should ensure the cost of the optimal investment portfolio of the new model is at least as low as the cost of the initial model. The process of generalizing carry-forward allowed cash-flow model starts with new assumptions and notations.

Let τ be the dimension of the liability vector $\boldsymbol{l}$ and assume that the asset cash-flow matrix C consists of τ columns. We define

$$\boldsymbol{g} = C^T n - l \tag{8}$$

where $\boldsymbol{g} = (g_1, g_2, ..., g_\tau)^T$

Let r_t denote a conservative estimate of the one-period reinvestment interest rate at time t, for $t = 1, 2, 3, ..., \tau - 1$

Let b_t denote the (cumulative) cash balance at time t, for $t = 1, 2, 3, ..., \tau$

We have $b_1 = g_1$ for $t = 1, 2, 3, ..., \tau - 1$

From the notations above, it is clear that

$$b_2 = g_2 + (1 + r_1)b_1$$

$$b_3 = g_3 + (1 + r_2)b_2$$

$$...$$

$$b_{t+1} = g_{t+1} + (1 + r_t)b_t \tag{9}$$

that is, the cash balance at time $t + 1$ is equal to the carry-forward amount of positive cash balance plus the accumulated value of cash balance at time t after one period $[t,t + 1]$.

Recall from the previous section that the business needs to find an investment portfolio $\{n_k|n_k \geq 0\}$, which minimizes the cost

$$\boldsymbol{p^T n} = \sum_{n=1}^{k} p_k n_k$$

while being constrained to the condition that the cash balances $\{b_t\}$ are to be non-negative

$$\{b_t\} \geq 0$$

Note that the formulation given in Section II.2 is a special case of the model above with $r_1 = r_2 = r_3 = ... = -1$. Which means $b_1 = g_1, b_2 = g_2, b_3 = g_3, ..., b_{t+1} = g_{t+1}$.

We now extend the model in the previous section by defining

$$R = \begin{pmatrix} -1 & 0 & 0 & . & . & 0 & 0 \\ 1+r_1 & -1 & 0 & . & . & 0 & 0 \\ 0 & 1+r_2 & -1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1+r_{\tau-1} & -1 \end{pmatrix} \tag{10}$$

and $\boldsymbol{b} = (b_1, b_2, ..., b_\tau)^T$, the generalized linear problem **LP2** is

$$\text{Minimize } \boldsymbol{p^T n} \tag{11}$$

$$\boldsymbol{n} \geq \boldsymbol{0}, \boldsymbol{b} \geq \boldsymbol{0}$$

subject to

$$\begin{pmatrix} \boldsymbol{C^T R} \end{pmatrix} \times \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix} = \boldsymbol{l} \tag{12}$$

where n and b are decision variables.

An alternative expression of **LP2** can be written as

$$\text{Minimize } \begin{pmatrix} p^T \ 0^T \end{pmatrix} \times \begin{pmatrix} n \\ b \end{pmatrix} \tag{13}$$

subject to

$$C^T n + Rb = l$$

$$\implies Rb + C^T n = 0$$

$$\implies Rb + g = 0 \text{ or } Rb = -g$$

$$\implies \begin{pmatrix} -1 & 0 & 0 & . & . & 0 & 0 \\ 1+r_1 & -1 & 0 & . & . & 0 & 0 \\ 0 & 1+r_2 & -1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1+r_{\tau-1} & -1 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ . \\ . \\ b_\tau \end{pmatrix} = \begin{pmatrix} -g_1 \\ -g_2 \\ -g_3 \\ . \\ . \\ . \\ -g_\tau \end{pmatrix}$$

$$\implies \begin{pmatrix} -b_1 \\ (1+r_1)b_1 - b_2 \\ (1+r_2)b_2 - b_3 \\ . \\ . \\ . \\ (1+r_{\tau-1})b_{\tau-1} - b_\tau \end{pmatrix} = \begin{pmatrix} -g_1 \\ -g_2 \\ -g_3 \\ . \\ . \\ . \\ -g_\tau \end{pmatrix}$$

$$\implies -g_{t+1} = (1+r_t)b_t - b_{t+1}$$

$$\implies b_{t+1} = g_{t+1} + (1+r_t)b_t \text{ , which is (9)}$$

Because (12) is an equality constraint, ***Theorem 5*** of the fundamental theorem of linear programming will be applied. The dual problem of **LP2** - denoted as **LP*2** is

$$\underset{\boldsymbol{v}}{\text{Maximize}} \ \boldsymbol{l^T v} \tag{14}$$

subject to

$$\begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{R^T} \end{pmatrix} \times \boldsymbol{v} \leq \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{0} \end{pmatrix} \tag{15}$$

Based on the fundamental theorem of linear programming, there is no explicit sign restriction on $\boldsymbol{v}$. However, in this section we will show that $\boldsymbol{v}$ has to be non-negative because the reinvestment interest rates are non-negative.

The constraint (15) is equivalent to the pair of matrix inequalities:

$$\boldsymbol{Cv} \leq \boldsymbol{p},$$

which is the same as inequality (4), and,

$$\boldsymbol{R^T v} \leq \boldsymbol{0}, \tag{16}$$

Break down the inequality (16), we have:

$$R^T v = v^T R \leq 0$$

$$\Longrightarrow \begin{pmatrix} v_1 & v_2 & v_3 & . & . & . & v_\tau \end{pmatrix} \times \begin{pmatrix} -1 & 0 & 0 & . & . & 0 & 0 \\ 1+r_1 & -1 & 0 & . & . & 0 & 0 \\ 0 & 1+r_2 & -1 & . & . & 0 & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & 1+r_{\tau-1} & -1 \end{pmatrix} \leq 0$$

$$\Longrightarrow \begin{pmatrix} -v_1 + (1+r_1)v_2 \\ -v_2 + (1+r_2)v_3 \\ -v_3 + (1+r_3)v_4 \\ . \\ . \\ -v_{\tau-1} + (1+r_{\tau-1})v_\tau \\ -v_\tau \end{pmatrix} \leq 0$$

Therefore, inequality (16) is equivalent to the system of linear inequalities:

$$(1+r_t)v_{t+1} \leq v_t \text{ , with } t = 1, 2, 3, ..., \tau - 1 \tag{17}$$

$$\text{and } v_\tau \geq 0 \tag{18}$$

For $\boldsymbol{v} = (v_1, v_2, v_3, ..., v_\tau)^T$, define

$$\phi_t = \frac{v_t}{v_{t+1}} - 1 \text{ , } t = 1, 2, 3, ..., \tau - 1 \tag{19}$$

Then (17) can be expressed as

$$(1+r_t)v_{t+1} \leq v_t$$
$$\Longrightarrow r_t \leq \frac{v_t}{v_{t+1}} - 1$$
$$\Longrightarrow r_t \leq \phi_t \text{ , } t = 1, 2, 3, ..., \tau - 1 \tag{20}$$

Since $\{r_t\}$ is the reinvestment rates, it should be non-negative. Recall that we have the monotonicity condition (7) from the previous section, which is $v_1 \geq v_2 \geq ... \geq v_t ... \geq 0$.

Recall that i_t denotes the t-period spot rate. For $t = 1, 2, 3, ...$, we have the formula of the one-period forward rate at time t - denoted as f_t

$$f_t = \frac{(1+i_{t+1})^{t+1}}{(1+i_t)^t} - 1 \tag{21}$$

f_t may be interpreted as the market forecast, at time 0, of the one-period interest rate at time t. The forward rate can be understood as an estimate of the interest rate from time t

to time $t+1$ based on today's information, and the reinvestment rate r_t is the rate of return that we expect to receive if we invest any surplus at time t. Based on the information the business have right now, we expect to earn a reinvestment rate $r_t = f_t$ from time t to time $t+1$. However, in reality, the reinvestment rate will likely to be lower than the forward rate. Therefore, when choosing a value for the reinvestment rate r_t, it is prudent to ensure that

$$0 \leq r_t \leq f_t \tag{22}$$

With the condition above, we need to verify that vector

$$\boldsymbol{v} = [(1+i_1)^{-1}, (1+i_2)^{-2}, ..., (1+i_t)^{-t}]$$

is a feasible solution of **LP*2**. It means we must verify that vector $\boldsymbol{v}$ satisfies the constraint (17). In fact, the condition (22) implies that vector $\boldsymbol{v}$ is actually a feasible solution with respect to **LP*2**, this can be proven as follow:
We have $v_t = (1+i_t)^{-t}$ and recall that condition (17) is $(1+r_t)v_{t+1} \leq v_t$, with $t = 1, 2, 3, ..., \tau - 1$ condition (17) is equivalent to

$$(1+r_t)(1+i_{t+1})^{-(t+1)} \leq (1+t_t)^{-t}$$
$$\Longrightarrow (1+r_t) \leq \frac{(1+i_{t+1})^{t+1}}{(1+i_t)^t}$$
$$\Longrightarrow r_t \leq f_t \ ,$$

which satisfies condition (22).
The condition (22) also implies that the present value of the liability cash flows (with respect to the current term structure of interest rates) is a lower bound for the cost of the optimal investment portfolio based on ***Theorem 3***.

As mentioned above in this section, the reinvestment rates $\{r_t\}$ should be non-negative, further more we have the condition (7), $v_1 \geq v_2 \geq ... \geq v_\tau \geq 0$. We now look back to the example at the end of Section III.1 to check if the monotonicity condition (7) is satisfied with the appearance of reinvestment rate $r_t = 0.05$. Indeed, the optimal solution of **LP*2** is $\boldsymbol{v} = (0.9709, 0.9246, 0.8806, 0.8387, 0)$, which satisfies condition (7). Furthermore, we notice that by moving from **LP1** to **LP2**, which is allowing the positive cash balances to be carried forward, the optimal value is lowered. The optimal value of **LP1** of the previous section was

19.3971, that of **LP2** is **16.0153** (Refer to Appendix A for more details). This fact is quite interesting since **LP*2** has an extra set of constraints not present in **LP*1**. What if we keep extend the model and impose further constraints on **LP*2**? The optimal value should even come down further. The next section will explain in details the formulation of the last model.

III.3 Borrowing also allowed in the Cash-flow matching model

The classical cash-flow matching model was improved by permitting the carry-forward of positive cash balances. Section III.2 showed that by allowing the positive cash balance at time $t = 1$ to be carried forward at time $t = 2$, the optimal value is lowered and the condition (7) for the feasible vector $\boldsymbol{v}$ is satisfied. From an economic point of view, maybe if we impose further constraints on the linear problem **LP*2** the optimal value should be even lower. Motivated by the constraint (20), we *might* impose extra conditions that

$$s_t \geq \phi_t \ , \ t = 1, 2, 3, ..., \tau - 1 \tag{23}$$

where $\{s_t\}$ are constants; that is, we consider the linear program formulation:

$$\text{Maximize } \boldsymbol{l^T v}$$

$$\boldsymbol{v}$$

subject to

$$\boldsymbol{Cv} \leq \boldsymbol{p}$$

$$r_t \leq \phi_t \leq s_t \ , \ t = 1, 2, 3, ..., \tau - 1 \tag{24}$$

and

$$0 \leq v_\tau$$

Recall that r_t is the one-period reinvestment rate at time t. Based on the formulation above, s_t might be be interpreted as the one-period borrowing rate at time t. This section will define s_t with the help of the fundamental theorem of linear programming. However, note that in order to avoid arbitrage opportunities, the interest rate for borrowing should be at least as high as the interest rate for investing, that is $s_t \leq r_t$. (The $\{s_t\}$ will turn out to satisfy condition (23)).

Motivated by Section III.2, for $t = 1, 2, 3, ..., \tau - 1$ let r_t and s_t denote the one-period reinvestment and borrowing rates respectively at time t. As we mentioned earlier in Section III.2, let $C^T\boldsymbol{n} - \boldsymbol{l} = (g_1, g_2, ..., g_t)^T$.

For $t = 1, 2, 3, ..., \tau$, let b_t denote the cumulative cash balance at time t.

We have $b_1 = g_1$, and for $t = 1, 2, 3, ..., \tau - 1$

$$b_{t+1} = g_{t+1} + (1 + r_t)b_t \quad \text{if } b_t \geq 0 \tag{25}$$

and

$$b_{t+1} = g_{t+1} + (1 + s_t)b_t \quad \text{if } b_t < 0 \tag{26}$$

Recall that the business needs to find an investment portfolio $\{n_k | n_k \geq 0\}$ which minimizes the cost

$$\boldsymbol{p^T n = \sum_{n=1}^{k} n_k p_k}$$

subject to the condition that the cash balance is non-negative

$$\{b_t\} \geq 0$$

Note that the formulation in Section III.2 is a special case of the formulation above with $s_1 = s_2 = s_3 = ... = +\infty$.

Since the business should not have riskless arbitrage opportunities, for each t, the condition $r_t \leq s_t$ must be imposed on this model. However, unless $r_t = s_t$ for all t, the program is non-linear, since (25) and (26) takes different values if r_t is not the same as s_t. This non-linear program can be converted into a linear one by extending **LP*2**.

We now extend the linear program **LP*2**. Let S denote the τ by $(\tau - 1)$ matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & . & . & 0 \\ -1 - s_1 & 1 & 0 & . & . & 0 \\ 0 & -1 - s_2 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & -1 - s_{\tau-1} \end{pmatrix} \tag{27}$$

consider the condition (23) mentioned above, we have the inequality $S^T\boldsymbol{v} \leq 0$

$$S^T v = v^T S \geq 0$$

$$\implies (v_1 \quad v_2 \quad v_3 \quad . \quad . \quad . \quad v_\tau) \times \begin{pmatrix} 1 & 0 & 0 & . & . & 0 \\ -1-s_1 & 1 & 0 & . & . & 0 \\ 0 & -1-s_2 & 1 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & -1-s_{\tau-1} \end{pmatrix} \leq 0$$

$$\implies \begin{pmatrix} v_1 - (1+s_1)v_2 \\ v_2 - (1+s_2)v_3 \\ v_3 - (1+s_3)v_4 \\ . \\ . \\ v_{\tau-2} - (1+s_{\tau-2})v_{\tau-1} \\ v_{\tau-1} - (1+s_{\tau-1})v_\tau \end{pmatrix} \leq 0$$

the inequality above is equivalent to the system of linear inequalities:

$$(1+s_t)v_{t+1} \geq v_t \text{ , with } t = 1, 2, 3, ..., \tau - 1 \tag{28}$$

$$\implies s_t \geq \frac{v_t}{v_{t+1}} - 1 \tag{29}$$

$$\implies s_t \geq \phi_t \tag{30}$$

We consider the following linear program - denoted as **LP*3**

$$\text{Maximize } \boldsymbol{l^T v} \tag{31}$$

$$\boldsymbol{v}$$

subject to

$$\begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{R^T} \\ \boldsymbol{S^T} \end{pmatrix} \times \boldsymbol{v} \leq \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix} \tag{32}$$

The problem dual to **LP*3** - denoted as **LP3** is

$$\text{Minimize } (\boldsymbol{p^T} \; \boldsymbol{0^T} \; \boldsymbol{0^T})y \geq 0 \tag{33}$$

subject to

$$(C^T \; R \; S)\boldsymbol{y} = \boldsymbol{l} \tag{34}$$

Naturally, the vector $\boldsymbol{y}$ has three components, according to (34), b^+ and b^- will match matrix R and S respectively

$$\boldsymbol{y} = \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{b}^+ \\ \boldsymbol{b}^- \end{pmatrix} \tag{35}$$

where

$$\boldsymbol{b}^+ = (b_1^+, b_2^+, b_3^+, ..., b_\tau^+)^T \tag{36}$$

$$\boldsymbol{b}^- = (b_1^-, b_2^-, b_3^-, ..., b_\tau^-)^T \tag{37}$$

LP3 can be rewritten as

$$\text{Minimize } \boldsymbol{n^T p} \tag{38}$$

$$\boldsymbol{n \geq 0, b^+ \geq 0, b^- \geq 0} \tag{39}$$

We now unscramble (34) in terms of $\{g_t\}$

$$(C^T\; R\; S)\boldsymbol{y} = \boldsymbol{l}$$

$$\implies C^T\boldsymbol{n} + R\boldsymbol{b}^+ + S\boldsymbol{b}^- = \boldsymbol{l}$$

$$\implies C^T\boldsymbol{n} - \boldsymbol{l} + R\boldsymbol{b}^+ + S\boldsymbol{b}^- = 0$$

$$\implies \boldsymbol{g} + R\boldsymbol{b}^+ + S\boldsymbol{b}^- = 0$$

Expanding the matrix equation in terms of components will yield

$$b_1^+ - b_1^- = g_1, \tag{40}$$

$$b_{t+1}^+ - b_{t+1}^- = g_{t+1} + (1 + r_1)b_1^+ + (1 + s_1)b_1^-, \tag{41}$$

$$\text{with } t = 1, 2, 3, ..., \tau - 2$$

and

$$b_\tau^+ = g_\tau + (1 + r_{\tau-1})b_{\tau-1}^+ + (1 + s_{\tau-1})b_{\tau-1}^- \tag{42}$$

The problem **LP3** is now a linear program, this is the desired linearization of the non-linear program at the beginning of this section. The problem is we can see that (41) is not equivalent to (25) and (26). Though if each s_t is greater than the corresponding r_t, when we found the feasible solution of the linear program, at most one of b_t^+ and b_t^- is nonzero for each $t = 1, 2, 3, ..., \tau - 1$. Therefore (41) is only equivalent to (25) and (26) at the feasible solution of the linear problem.

We now go back to the example at the end of section III.1. A new set of constraint on the borrowing rate s_t was introduced. Solving **LP*3** with the data given in the example, we got the optimal value of **13.966**, which is the lowest amongst the three models and the closet to the present value of the liability cash flow. Furthermore, the feasible solution $\boldsymbol{v} = (0.9709, 0.9246, 0.8806, 0.8387, 0.7357)$, which satisfies monotonicity condition (7). For more details on the calculation process, please refer to Appendix A.

The development of the cash-flow matching model has progressed through three distinct stages. The initial stage involved the classical cash-flow matching model, where meticulous efforts were made to achieve a perfect alignment between cash inflows and outflows. However, due to the non-fulfillment of certain model conditions, we proceeded to the next stage. In the next stage, we allow the act of carrying forward a positive cash balance. Through this modification, we observed a discernible pattern, prompting us to expand the model further by allowing both carry-forward and borrowing. It is important to note that while these advancements have yielded improved outcomes, they do carry an element of risk. The act of reinvesting and borrowing may deviate from initial expectations, necessitating careful consideration of associated risks.

To analyze the capability of the cash-flow matching model. Section IV. Applications of the Cash-flow matching model will demonstrate how this model can be applied in real-world scenario.

APPLICATIONS OF THE CASH-FLOW MATCHING MODEL

IV.1 The inspiration of Model application

My journey through the Faculty of Mathematical Economics at the National Economics University has been profoundly influential in shaping the research I am undertaking. The solid foundation in mathematics and data analytics that I acquired during my studies has been instrumental in the development of this thesis. I draw inspiration from two specific subjects: Investment and Financial Markets, taught by Mrs. Pham Thi Hong Tham, and Quantitative Risk Management, instructed by Mr. Hoang Duc Manh and Mrs. Nguyen Thi Lien. In these courses, I delved into the optimization of bond portfolios for investment purposes, albeit with a primary focus on minimizing risk and cost.

Building upon the knowledge acquired from past lectures, a compelling idea took shape: the creation of a portfolio designed to shield businesses from the risk of default. By strategically selecting a diverse range of bonds and developing a cash-flow model with linear programming techniques, the objective is to establish a portfolio that can withstand interest rate risk and ultimately enhance the business's ability to meet its financial obligations without succumbing to default.

IV.2 Data descriptions

IV.2.1 The cash flow of Liability obligations

The cash-flow matching model has been formulated in the previous sections. We can see that by taking the dual of a linear program, a problem can be viewed from a different perspective. By formulating the cash-flow matching model, it is discovered that the ratios of the discount factors are related to the forward rates; and bounding these rates from above and below leads to an improved model.

In this section, we will bring the model in to real-life applications, the data gathering process, calculating procedures will be described in details. The cash-flow matching model is applied on the data of Vietnam Military Bank (MBBank) liability obligations with the purchase of Vietnam Treasury Bonds. First of all, to gather MB Bank's liability stream data, MB Bank's official website and the sections that contain information about the bank's liabilities was explored. Every year, MB Bank publish their 'Annual Financial Reports' with details of liability obligation through out the year. Collecting those information in 10-year time period (from 2013 to 2022), MB Bank liability stream was formed:

Year	**Total Liability**	**Liability Growth rate**
2013	VND 163,809,436	1.11
2014	VND 182,141,432	1.08
2015	VND 196,405,769	1.15
2016	VND 225,093,073	1.24
2017	VND 278,545,471	1.15
2018	VND 320,276,725	1.13
2019	VND 361,280,478	1.19
2020	VND 431,103,551	1.21
2021	VND 519,690,754	1.19
2022	VND 618,064,357	

Table IV.2.1 : MB Bank liability cash stream (*Unit: million VND*)

Source: mbbank.com.vn/Investor/bao-cao-tai-chinh/

Since the data is collected in the 'Annual Financial Report' of MB Bank, we will assume the liability obligations must be paid at the end of a fiscal year. The fiscal year of MB Bank is from June 1st to May 31st of the following year. Therefore, liabilities must be paid at May 31st every year. We also notice the appearance of the Liability Growth Rate column. Due to the fact that the cash-flow model is used for future cash outflows, MB Bank's future liability stream must be forecast. In the endeavor to forecast these future liabilities, numerous quantitative forecasting tools are commonly employed. However, in this particular study, we employ the straight-line forecast method as our chosen approach for predicting future liabilities.

The straight-line forecast method stands as a straightforward and user-friendly forecasting technique. Employing historical data and trends, financial analysts utilize this method to project future liability growth. Interpreted from the historical data, we have the trend of liability stream from year 2013 to 2022 shown in the graph below:

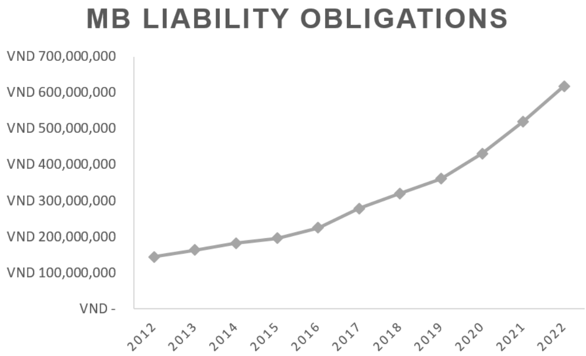


Figure IV.2.1 : MB Bank's liability obligations from 2013 to 2022

It is noticeable that the Total liability increases every year with the average liability growth rate equals to **1.16** (according to the data in Table 3.). From that we will assume the growth rate will remain constant into the future, the same rate **1.16** will be used for 2023-2032 time period. To forecast future liabilities, we take previous year's figure and multiply it by the average growth rate. The result of future liability cash flow will be shown in Table 4.

Date	Total Liability
31/05/2023	VND 716,930,379
31/05/2024	VND 831,611,081
31/05/2025	VND 964,636,191
31/05/2026	VND 1,118,940,095
31/05/2027	VND 1,297,926,562
31/05/2028	VND 1,505,543,834
31/05/2029	VND 1,746,371,715
31/05/2030	VND 2,025,722,599
31/05/2031	VND 2,349,758,651
31/05/2032	VND 2,725,627,746

Table IV.2.2 : Forecast of MB Bank liability cash flows (*Unit: million VND*)

IV.2.2 Data of Government Treasury Bonds

Having obtained MB Bank's projected liability stream, our next step involves applying the model to align cash outflows with cash inflows. Even though the bond coupon of Treasury Bond is not the best, it has minimal risk which can help businesses avoid being default. This model can be applied to any type of bonds available on the market, depends on the risk appetite of each business. This paper specifically considers risk-averse businesses and explores the option of acquiring Government Treasury Bonds as a means of mitigating potential risks. To facilitate this analysis, we retrieve information regarding bond types, bond prices, and bond maturities from the Hanoi Stock Exchange database.

In analyzing the original Treasury Bond data, it is worth noting that each bond carries a distinct issue date. However, for the purpose of our analysis, we will make the assumption that all bonds are annual and they were uniformly issued on a single date. This assumption effectively eliminates any time gap between cash inflows and cash outflows, simplifying the cash flow matching process and allowing for a more streamlined analysis of the data. Generally, the database encompasses ten distinct bonds with varying maturity periods:

Bonds	Bond Prices	Coupon	Issue date	Maturity date
TPCP 1y	102,839	5.3%	01/06/2023	31/05/2023
TPCP 2y	98,954	1.9%	01/06/2023	31/05/2024
TPCP 3y	104,741	4.3%	01/06/2023	31/05/2025
TPCP 4y	100,538	2.8%	01/06/2023	31/05/2026
TPCP 5y	105,082	3.6%	01/06/2023	31/05/2027
TPCP 6y	119,374	1.9%	01/06/2023	31/05/2028
TPCP 7y	129,295	7.5%	01/06/2023	31/05/2029
TPCP 8y	97,382	3.7%	01/06/2023	31/05/2030
TPCP 9y	95,200	2.2%	01/06/2023	31/05/2031
TPCP 10y	113,293	4.8%	01/06/2023	31/05/2032

Table IV.2.3 : Government Treasury Bond Data (*Unit: thousand VND*)

Source: https://www.hnx.vn/vi-vn/trai-phieu.html

From the data given above, the cash-flow matrix - denoted as $\boldsymbol{C}$ was formed

Bonds	CF at time 1	CF at time 2	CF at time 3	CF at time 4	CF at time 5	CF at time 6	CF at time 7	CF at time 8	CF at time 9	CF at time 10
TPCP 1y	108289.467	0	0	0	0	0	0	0	0	0
TPCP 2y	1880.126	100834.126	0	0	0	0	0	0	0	0
TPCP 3y	4503.863	4503.863	109244.863	0	0	0	0	0	0	0
TPCP 4y	2815.064	2815.064	2815.064	103353.064	0	0	0	0	0	0
TPCP 5y	3782.952	3782.952	3782.952	3782.952	108864.952	0	0	0	0	0
TPCP 6y	2268.106	2268.106	2268.106	2268.106	2268.106	121642.106	0	0	0	0
TPCP 7y	9697.125	9697.125	9697.125	9697.125	9697.125	9697.125	138992.125	0	0	0
TPCP 8y	3603.134	3603.134	3603.134	3603.134	3603.134	3603.134	3603.134	100985.134	0	0
TPCP 9y	2094.4	2094.4	2094.4	2094.4	2094.4	2094.4	2094.4	2094.4	97294.4	0
TPCP 10y	5438.064	5438.064	5438.064	5438.064	5438.064	5438.064	5438.064	5438.064	5438.064	118731.064

Table IV.2.4 : Cash-flow of Treasury Bonds *(Unit: thousand VND)*

To facilitate the calculation process, we obtain the transpose matrix of $\boldsymbol{C}$

108289.467	1880.126	4503.863	2815.064	3782.952	2268.106	9697.125	3603.134	2094.4	5438.064
0	100834.126	4503.863	2815.064	3782.952	2268.106	9697.125	3603.134	2094.4	5438.064
0	0	109244.863	2815.064	3782.952	2268.106	9697.125	3603.134	2094.4	5438.064
0	0	0	103353.064	3782.952	2268.106	9697.125	3603.134	2094.4	5438.064
0	0	0	0	108864.952	2268.106	9697.125	3603.134	2094.4	5438.064
0	0	0	0	0	121642.106	9697.125	3603.134	2094.4	5438.064
0	0	0	0	0	0	138992.125	3603.134	2094.4	5438.064
0	0	0	0	0	0	0	100985.134	2094.4	5438.064
0	0	0	0	0	0	0	0	97294.4	5438.064
0	0	0	0	0	0	0	0	0	118731.064

Table IV.2.5 : The transpose matrix of Cash-flow of Treasury Bonds *(Unit: thousand VND)*

With the comprehensive collection of essential data for this model, encompassing Liability Cash flows, Bond types, Bond prices, and Bond Cash-flow matrix, we find ourselves on the verge of completing the necessary information. The remaining pieces of the problem lie in determining the reinvestment rate and the borrowing rate, which vary depending on the specific business and its risk appetite. However, to proceed with our analysis, we adopt a generalized assumption of a $\boldsymbol{r_t = 2\%}$ reinvestment rate and a $\boldsymbol{s_t = 10\%}$ borrowing rate. While these rates may vary in reality, this assumption provides us with a reasonable starting point for our cash flow matching analysis.

Continuing with the calculating process, we delve into the heart of the model and unveil the results it produces. Through meticulous analysis and application of the established methodology, we extract valuable insights and outcomes from the model. These results shed light on the alignment of cash outflows and inflows, providing a comprehensive understanding of the effectiveness and efficiency of the cash flow matching approach.

IV.3 Model results

In this section, we apply the formulated model from the preceding sections to determine the optimal value of the Bond portfolios. The outcomes of three distinct models, namely LP1, LP2, and LP3, will be presented, allowing for a comprehensive comparison of the results obtained from each model. For a comprehensive breakdown of the computational process, please consult Appendix B for detailed information.

It is important to note that when predicting future liability obligations, there is insufficient data available to determine the cash balances, whether they are positive or negative. As a result, it becomes necessary to first solve the dual problem in order to determine the optimal value. According to the fundamental theorem of linear programming duality (***Theorem 4***), the optimal value of the primal problem is equivalent to that of the dual problem. Therefore, to ascertain the quantity of bonds to be purchased, we must rely on the outcome of the primal problem.

Foremost, we obtain the Present value of the liability cash flow under the current term structure of interest rates (refer to Appendix B for details):

$$\boldsymbol{l^T C^{-1} p} = 10,072,237,520.9$$

IV.3.1 The Classical model - LP1

The linear program **LP1** is as follow:

$$\text{Minimize } \boldsymbol{p^T n}$$

subject to

$$\boldsymbol{C^T n} \geq \boldsymbol{l}$$

and the dual program **LP*1**

$$\text{Maximize } \boldsymbol{l^T v}$$
$$\boldsymbol{v} \geq \boldsymbol{0}$$

subject to

$$\boldsymbol{Cv} \leq \boldsymbol{p}$$

Through the calculation process, we get the Optimal value of **LP*1** is

$$\boldsymbol{l^T v} = 11,872,237,520.94$$

The obtained result aligns with our initial expectations, as the primary objective of the model was to achieve a perfect match between cash inflows and outflows. By striving for this precise alignment, the model aims to optimize cash flow management and mitigate any potential gaps or imbalances. Therefore, the realization of this expected outcome reinforces the effectiveness of the model in attaining its intended objective of precise cash flow matching.
The Feasible solution of **LP*1** is

$$\boldsymbol{v} = (0.9497, 0.9636, 0.8799, 0.8967, 0.8370, 0.8969, 0.5518, 0.7511, 0.8337, 0.6079)$$

From that, based on ***Theorem 4***, we have the Optimal value of **LP1** is

$$\boldsymbol{p^T n} = 11,872,237,520.94$$

applied the constraints of the linear program **LP1**, the Optimal Bond portfolio 1 is

$$\boldsymbol{n} = (2487.55, 3881.16, 5006.43, 6974.78, 8563.33, 9548.67, 10846.12,$$
$$18349.14, 22867.92, 22956.32)$$

The Optimal Bond Portfolio presented here achieves a perfect match between liabilities and bond payments. It is worth noting that this portfolio is comparatively more expensive than the ones discussed subsequently. A table comparing the prices of the portfolios and the corresponding number of bonds purchased will be provided at the end of this section for further analysis and comparison.

IV.3.2 The Carry-forward allowed model - LP2

The linear program **LP2** is as follow:

$$\text{Minimize } \boldsymbol{p^T n}$$

$$\boldsymbol{n} \geq \mathbf{0}, \boldsymbol{b} \geq \mathbf{0}$$

subject to

$$\begin{pmatrix} \boldsymbol{C^T R} \end{pmatrix} \times \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix} = \boldsymbol{l}$$

and the dual program **LP*2**

$$\text{Maximize } \boldsymbol{l^T v}$$

$$\boldsymbol{v}$$

subject to

$$\begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{R^T} \end{pmatrix} \times \boldsymbol{v} \leq \begin{pmatrix} \boldsymbol{p} \\ \mathbf{0} \end{pmatrix}$$

Through the calculation process, we get the Optimal value of **LP*2** is

$$\boldsymbol{l^T v} = 10,404,335,704.73$$

We notice that in this model, by allowing positive cash balances to be carried forward, the Optimal value has decreased and the Feasible solution of **LP*2** is

$$\boldsymbol{v} = (0.9497, 0.9310, 0.8812, 0.8640, 0.8393, 0.8228, 0.5613, 0.5503, 0.5395, 0.5289)$$

Vector $\boldsymbol{v}$ satisfies the monotonicity condition of the model. And as same as above, we apply the fundamental theorem of linear programming duality and the constraints of **LP2**, the Optimal Bond portfolio 2 is

$$\boldsymbol{n} = (3264.73, 4731.66, 5825.21, 7864.47, 9438.37, 10346.69, 11596.90,$$
$$19420.72, 24004.62, 2619.26)$$

IV.3.3 The Carry-forward and Borrowing allowed model - LP3

The linear program **LP3** is as follow:

$$\text{Minimize } (\boldsymbol{p^T 0^T 0^T})y \geq 0$$

subject to

$$(C^T \, R \, S)\boldsymbol{y} = \boldsymbol{l}$$

and the dual program **LP*3**

$$\text{Maximize } \boldsymbol{l^T v}$$

$$\boldsymbol{v}$$

subject to

$$\begin{pmatrix} \boldsymbol{C} \\ \boldsymbol{R^T} \\ \boldsymbol{S^T} \end{pmatrix} \times \boldsymbol{v} \leq \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{pmatrix}$$

Through the calculation process, we get the Optimal value of **LP*3** is

$$\boldsymbol{l^T v} = 10,108,635,497.07$$

The Feasible solution of **LP*3** is

$$\boldsymbol{v} = (0.9497, 0.9310, 0.8626, 0.7842, 0.7129, 0.6481, 0.5892, 0.5776, 0.5663, 0.5552)$$

We now also apply the fundamental theorem of linear programming duality and the constraints of **LP3** to find the Optimal Bond portfolio 3

$$\boldsymbol{n} = (2814.61, 4239.07, 5350.99, 7349.18, 8931.57, 9884.50, 11162.06,$$
$$18800.08, 1124.84, 22956.32)$$

To provide a concise summary of the results, a table has been constructed to present the number of different bonds and their corresponding prices. This tabular representation allows for a clear and organized comparison of the bond portfolio. By displaying the quantity of each bond and its associated price, readers can easily understand the composition and valuation of the portfolio. The table serves as a visual aid to facilitate the interpretation of the results, enabling a quick assessment of the bond holdings. With this table, the key information is condensed and presented in a structured format, enhancing the accessibility and comprehensibility of the summary. The table of Optimal Bond portfolios and the table of Optimal value of three models will be displayed below:

Bonds	Model LP1	Model LP2	Model LP3
TPCP 1y	2488	3265	2815
TPCP 2y	3881	4732	4239
TPCP 3y	5006	5825	5351
TPCP 4y	6975	7864	7349
TPCP 5y	8563	9438	8932
TPCP 6y	9549	10347	9885
TPCP 7y	10846	11597	11162
TPCP 8y	18349	19421	18800
TPCP 9y	22868	24005	1125
TPCP 10y	22956	2619	22956

Table IV.3.1 : The Optimal Bond portfolio of three Cash-flow matching model

(Rounded up to the nearest whole number)

	Model LP1	Model LP2	Model LP3
Optimal value	11,872,237,520.94	10,404,335,704.73	10,108,635,497.07

Table IV.3.2 : The Optimal value of three Cash-flow matching model

(*Unit: million VND*)

We have analyzed three different bond portfolios, each with its own set of prices and associated risks. Starting with model LP1, it has the highest price but is preferred by risk-averse businesses due to its absence of reinvestment or borrowing risk.

Model LP2, on the other hand, allows positive cash balances to be carried forward, resulting

in a lower portfolio price. However, this model introduces the risk of reinvestment, as the actual reinvestment rate may not align with expectations, potentially impacting the ability to meet debt obligations.
Lastly, model LP3 yields the least expensive bond portfolio. However, it is important to note that this model carries the highest level of risk among the three, as it encompasses both reinvestment and borrowing rate risks. Borrowing rate risk arises when borrowing is permitted in the presence of negative cash balances, potentially incurring additional liability obligations.

Consequently, businesses can select the most suitable cash-flow matching model based on their risk appetite. This choice allows them to effectively manage their debt obligations while fostering business growth. By carefully considering the risk factors associated with each model and aligning them with their specific risk tolerance, businesses can make an informed decision. The chosen model will facilitate the optimal matching of cash inflows and outflows, ensuring timely payment of debts and providing a solid foundation for sustainable business expansion. Ultimately, the selection of the appropriate cash-flow matching model enables businesses to mitigate risk, enhance financial stability, and pursue their growth objectives with confidence.

CONCLUSION AND RECOMMENDATION

This study has delved into the application of cash flow matching models and their significance in managing liabilities and optimizing cash flow strategies. By analyzing the data of MB Bank's liability cash flow and incorporating the purchase of Vietnam Treasury Bonds, we have demonstrated the potential benefits of cash flow matching in reducing cash flow mismatches and mitigating financial risks.

Based on the findings of this research, I would like to offer the following recommendations:

Firstly, Cash Flow Matching strategies should be implemented in businesses. Businesses should consider adopting cash flow matching strategies to align their cash inflows and outflows more effectively. This can help them manage their liabilities, improve cash flow stability, and reduce the risk of liquidity.

Secondly, Businesses should customize their Reinvestment and Borrowing Rate. While this study assumed a specific reinvestment rate and borrowing rate, it is essential for businesses to evaluate and customize these rates based on their specific risk appetite and market conditions. Conducting thorough risk assessments and considering various economic factors can help determine optimal reinvestment and borrowing rates.

Lastly, continuously monitor and update the Model: The cash flow matching model should be reviewed, updated, and refined periodically to reflect changing market dynamics and business requirements. Regularly incorporating new data and adjusting the model parameters will enhance its effectiveness and relevance over time.

By implementing these recommendations, businesses can enhance their cash flow management practices, minimize risks, and improve overall financial performance, leading to greater stability and success in the long run.

APPENDIX A: The calculation process of Section III Examples

In this model, Microsoft Excel is the primary calculation method utilized. However, for Section III Examples, we only consider solving the Dual problem of the Linear program. To facilitate the calculations, three matrices were incorporated into Excel: the liability matrix (denoted as $\boldsymbol{l}$), the bond price matrix (denoted as $\boldsymbol{p}$), and the bond cash flow matrix (denoted as $\boldsymbol{C}$). By inputting these matrices into Excel, the necessary data for the analysis is organized and structured in a convenient manner. This enables efficient calculations and allows for the application of various formulas and functions available within the Excel software. By utilizing Excel's Solver function, the model can process the data effectively and generate the desired outcomes. We input in Excel the matrices respectively:

A	B	C	D	E	F
Matrix					
$l =$	7		$p =$	1	
	-3			1	
	6			1	
	8			1	
	-4			1	
$C^T =$	1.03	0.035	0.04	0.0425	0.045
	0	1.035	0.04	0.0425	0.045
	0	0	1.04	0.0425	0.045
	0	0	0	1.0425	0.045
	0	0	0	0	1.045

Note that we are using C^T since vector $\boldsymbol{v}$ is being displayed as $\boldsymbol{v^T}$ in the Excel file.

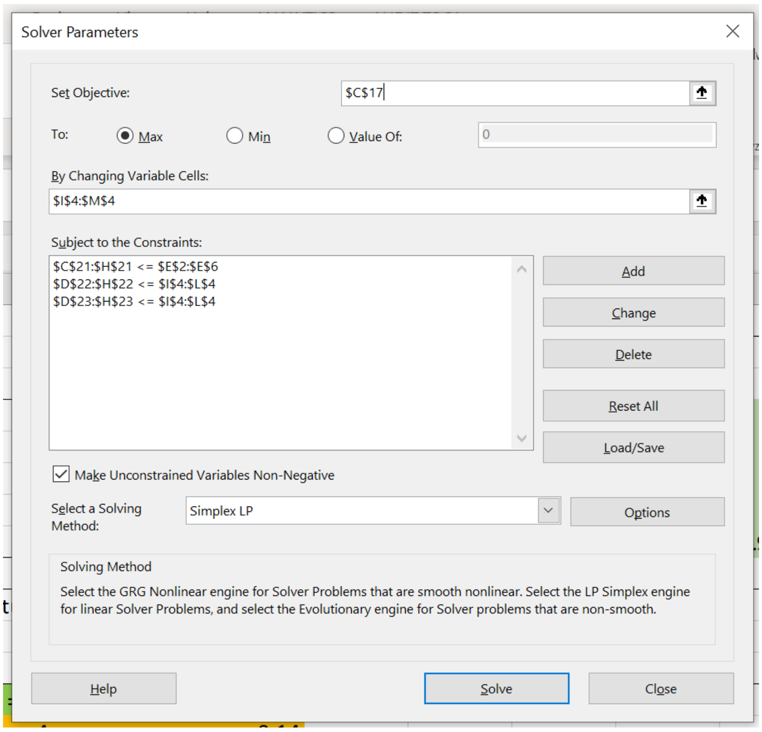
Next step is to calculate the Present value of the liability cash flow under the current term structure of interest rates. The Present value is computed by multiplying the liabilities with the inverse matrix of C^T multiply by the bond prices. The Excel function used for matrix multiplying was '=MMULT()'. The result of Present value of the liability cash flow is:

$l^T \times C^{-1} \times p =$	12.8878

From that, we input the linear program's objective function and constraints into Excel

Maximize			r1 = r2 = r3 = r4 =		0.05
$l^T \times v =$	13.966		s1 = s2 = s3 = s4 =		0.14
$v \geq 0$					
Constraints					
$C \times v \leq p$	1	0.91543	0.84984	0.79237	0.14512
$R^T \times v \leq 0$	1.01942	0.89423	0.78441	0.68808	0
$S^T \times v \leq 0$	1.1068	0.97087	0.85164	0.74706	0

Open the Solver add-in in Excel. In Excel 2016 or later versions, go to the "Data" tab, click on "Solver" in the "Analysis" group, and the Solver Parameters dialog box will appear.
In the Solver Parameters dialog box, set the objective function to be maximized or minimized by selecting the appropriate cell reference. Specify the decision variable cells to be adjusted by Solver.
Define the constraints by adding them one by one. Specify the constraint cells and their relationship to the constraint equation (e.g., less than or equal to, greater than or equal to, etc.). In this program, the Solver dialog box will look like this



We choose the solving method by selecting "Simplex LP" since the problem is linear. Click the "Solve" button to let Solver find the optimal solution. Solver will adjust the decision variable values to optimize the objective function while satisfying the constraints. The desired outcomes is the Optimal value and vector $\boldsymbol{v}$ shown in the paper. The maximum values of **LP*1**, **LP*2** and **LP*3** are

	LP*1	LP*2	LP*3
Optimal value	**19.3971**	**16.0153**	**13.966**

and the feasible vectors of problem **LP*1**, **LP*2** and **LP*3** respectively are

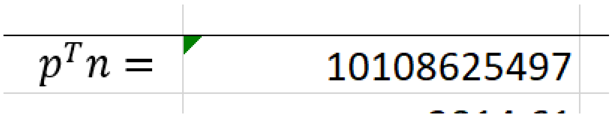
$v_{LP1} =$	**0.97087**	**0**	**0.9242**	**0.88198**	**0**
$v_{LP2} =$	**0.97087**	**0.92464**	**0.88061**	**0.83868**	**0**
$v_{LP3} =$	**0.97087**	**0.85164**	**0.74706**	**0.65531**	**0**

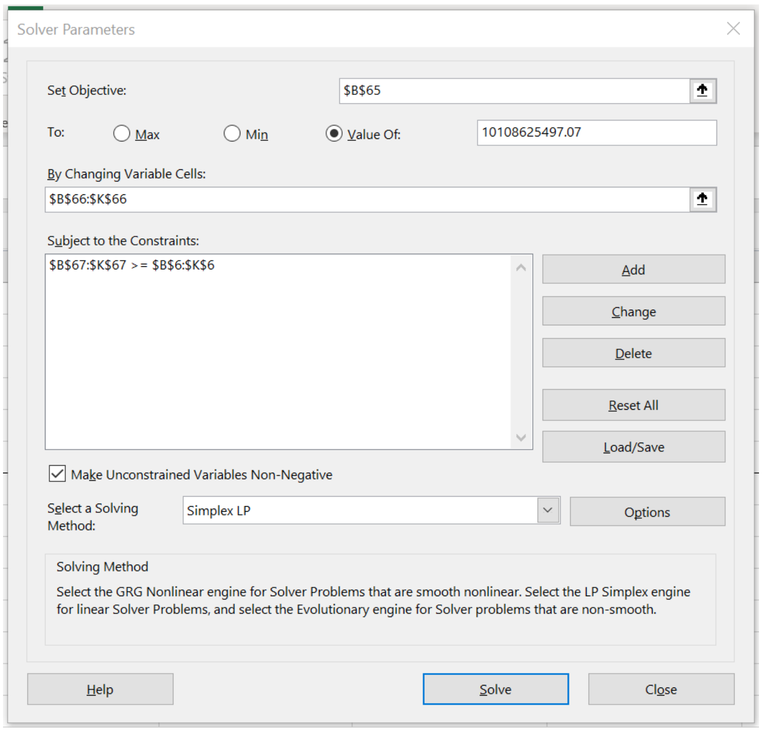
By using Excel Solver, we can efficiently solve linear programming problems and obtain optimal solutions for decision-making.

APPENDIX B: The calculation process of Section IV

The computational process for solving the Dual problem has been presented in Appendix A. Additionally, in Section IV, we carried out the calculation process, which involved solving the Primal problem as well. Appendix B will specifically focus on the procedure employed to calculate the number of Bonds purchased for each model.

As shown above, we apply the same method of adding the liability matrix, bond prices matrix and the bond cash flow matrix. Since the optimal value of primal problem is equal to the optimal value of dual problem, we set the objective function in the Solver dialog box to 'Value of'.





Note that we must specify the cell that we want to find the value for by selecting it in the "Set Objective" field. This is the cell whose value we want to optimize or find.

Click the "Solve" button to let Solver find the optimal value for the specified cell. Then review the Solver Results dialog box to see if Solver was able to find a solution. It will provide information on the optimal objective value and the variable values that achieve this optimum. Click "OK" to accept the solution and close the Solver Results dialog box. The value of the specified cell will be updated based on the optimal solution found by Solver.

The detailed results have been presented in Section IV, providing a comprehensive overview of the findings. Section IV encompasses a thorough examination of the outcomes, offering in-depth insights and analysis of the obtained results.

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